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THE HANKEL DETERMINANT $H_3(2)$ FOR SOME SPECIAL BOUNDED TURNING FUNCTIONS

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Abstract. In this paper, we determine the bound on the $H_3(2)$ Hankel determinant for some special functions of bounded turning in the unit disk.

1. Introduction

Let $\mathbb{A}$ denote the class of functions:

$$f(z) = z + a_2 z^2 + \cdots \tag{1.1}$$

which are analytic in the unit disk $E = \{z \in \mathbb{C}\colon |z| < 1\}$. Let $f, g \in \mathbb{A}$, we say that $f(z)$ is subordinate to $g(z)$ (written as $f \prec g$) if there exists a holomorphic function $w(z)$ (not necessarily schlict) in E, satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $z \in E$. The function $w(z)$ is known as a Schwarz function and has series expansion:

$$w(z) = c_1 z + c_2 z^2 + \cdots$$

where the coefficients c_k are complex constants.

The well-known traditional class $\mathbb{R}$ of bounded turning functions consisting of analytic functions $f(z)$ whose derivatives $f'(z)$ are subordinate to the Moebious function $L_0(z) = (1+z)/(1-z)$. That is, an analytic function $f(z)$ is said to be of bounded turning if and only if

$$f'(z) \prec \frac{1+z}{1-z}.$$

Equivalently, $f \in \mathbb{R}$ if and only if

$$f'(z) = \frac{1+w(z)}{1-w(z)}$$

for some Schwarz function $w(z)$. Equivalently also, $f \in \mathbb{R}$ if and only if $f(0) = 1$ and Re $f'(z) > 0$. Functions in $\mathbb{R}$ are generally univalent in the unit disk.

In this work, we consider the following special type of bounded turning functions consisting of analytic functions $f(z)$ satisfying:

$$f'(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2.$$

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First, we note that the real part of the superordinate function

$$g(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$$

which is Re $g(z) = 1 + \frac{4}{3}x + \frac{2}{3}(x^2 - y^2)$, satisfies

$$\text{Re } g(z) > \frac{4}{3}x^2 + \frac{4}{3}x + \frac{1}{3} > 0$$

for all $z \in E$. Hence it can be easily seen that $g(z)$ satisfies $g(0) = 1$ and Re $g(z) > 0$ and so, it follows that these functions are analytic in the unit disk and they form a subclass, denoted by $\mathcal{R}_s$, of the class $\mathcal{R}$, of bounded turning functions and are also generally univalent in the unit disk. They have been mentioned to be associated with certain heart-shape regions of the plane known as cardoids (see [2, 5]).

Our main task in this paper is the determination of the best possible upper bound on the $H_3(2)$ Hankel determinant for functions in $\mathbb{R}_s$. The Hankel determinant for univalent functions was defined much earlier by Noonan and Thomas in [3] as follows: let $n \geq 0$ and $q \geq 1$, the q-th Hankel determinant is defined as:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

The determinant, for specific parameters, $n \geq 0$ and $q \geq 1$, has continued to be investigated by researchers in the field up to this date with new techniques being developed to hit the best possible estimates for various class of functions including those associated with certain special geometries such as the lemniscate of Bernoulli, the cardiod and conic regions (see [2, 4, 5]). In particular, the first work on the third Hankel determinant was presented by Babalola [1] in 2010. Many more outstanding works have appeared in reputable journals following his pioneering work on the third Hankel determinant. References for many of such works can be found in [2, 4, 5] and we do not bother reciting them in this paper.

The $H_3(2)$ Hankel determinant for univalent functions is given by:

$$H_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ a_4 & a_5 & a_6 \end{vmatrix}$$

which is equivalent to:

$$H_3(2) = a_4(a_3a_5 - a_4^2) + a_5(a_3a_4 - a_2a_5) + a_6(a_2a_4 - a_3^2). \tag{1.1}$$

The Hankel determinants provide important tools in engineering and mathematical analysis. They have a wide range of applications, for example, in signal processing, control theory, orthogonal polynomials, moment problems, linear systems, sequences and series, and have received a lot of attention of researchers with new evaluation methods emerging from time to time.

In Section 2, we state the inequalities we depend on in this work. The inequalities are largely contained in a recent very comprehensive work of Zaprawa [6]. The work of Zaprawa undoubtedly provides a fountain of applicable inequalities for many future attempts in this interesting field.

We state and prove our main results in Section 3.

2. Preliminary Lemmas

Lemma 2.1. [6] *Let $w(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then for any real numbers μ and v the following inequalities hold:*

$$|c_k| \leq 1,\ k = 1, 2. \cdots, \tag{2.1}$$

$$|c_k - \mu c_1 c_{k-1}| \leq \max\{1, |\mu|\},\ \mu \in \mathcal{R},\ k = 2, 3, \cdots, \tag{2.2}$$

$$|c_{2k} \pm c_k^2| \leq 1,\ k = 1, 2, \cdots, \tag{2.3}$$

$$|c_{k-1}c_{k+1} - c_k^2| \leq 1,\ k = 2, 3, \cdots, \tag{2.4}$$

$$|c_2| \leq 1 - |c_1|^2, \tag{2.5}$$

$$|c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|} \tag{2.6}$$

and

$$|c_4| \leq 1 - |c_1|^2 - |c_2|^2. \tag{2.7}$$

Lemma 2.2. *Let $w(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then*

$$|c_3 \pm c_1^3| \leq 1 \qquad (2.8).$$

Proof. Since $|c_3 \pm c_1^3| \leq |c_3| + |c_1|^3$, then using (2.6), we have

$$|c_3 \pm c_1^3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|} + |c_1|^3.$$

Setting $x = |c_1|$ and $y = |c_2|$ we have:

$$|c_3 \pm c_1^3| \leq 1 - x^2 - \frac{y^2}{1 + x} + x^3 = h(x, y)$$

Noting $y = |c_2| \geq 0$, we have

$$h(x, y) \leq h(x, 0) = 1 - x^2 + x^3.$$

Noting also that $x = |c_1| \geq 0$, we find that the maximum of $h(x, 0)$ is at $x = 0$ and its maximum value is 1. Hence we have $|c_3 \pm c_1^3| \leq h(x, 0) \leq h(0, 0) = 1$ as required. □

3. Main Result

Theorem 3.1. *Let $f \in \mathbb{R}_s$. Then*

$$|a_2| \leq \frac{2}{3}, \quad |a_3| \leq \frac{4}{9}, \quad |a_4| \leq \frac{1}{3}, \quad |a_5| \leq \frac{2}{5}, \quad |a_6| \leq \frac{4}{9},$$

$$|a_3a_5 - a_4^2| \leq \frac{31}{135},$$

$$|a_3a_4 - a_2a_5| \leq \frac{52}{135}$$

and

$$|a_2a_4 - a_3^2| \leq \frac{20}{81}.$$

Furthermore

$$|H_3(2)| \leq \frac{6203}{18225} = 0.340356652949246.$$

These inequalities are the best possible.

Proof. Let $f \in \mathbb{R}_s$. Then by definition

$$f'(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2.$$

Then there exists a Schwarz function $w(z) = c_1z + c_2z^2 + \cdots$ such that

$$f'(z) = 1 + \frac{4}{3}w(z) + \frac{2}{3}w(z)^2. \tag{3.1}$$

Then the right-hand side of (3.1) gives series expansion:

$$1 + \frac{4}{3}c_1z + \frac{2}{3}(2c_2 + c_1^2)z^2 + \frac{4}{3}(c_3 + c_1c_2)z^3 + \frac{2}{3}(2c_4 + c_2^2 + 2c_1c_3)z^4 + \frac{4}{3}(c_5 + c_1c_4 + c_2c_3)z^5 + \cdots$$

Expanding left-hand side of (3.1) in series form and comparing coefficients, we find that:

$$a_2 = \frac{2}{3}c_1 \tag{3.2}$$

$$a_3 = \frac{2}{9}(c_1^2 + 2c_2) \tag{3.3}$$

$$a_4 = \frac{1}{3}(c_1c_2 + c_3) \tag{3.4}$$

$$a_5 = \frac{2}{15}(2c_1c_3 + c_2^2 + 2c_4) \tag{3.5}$$

$$a_6 = \frac{2}{9}(c_1c_4 + c_2c_3 + c_5). \tag{3.6}$$

First, we determine bounds on the coefficients from (3.2) to (3.6). As for the bound on a_2, it is easy to see from $|c_1| \leq 1$. We write

$$a_3 = \frac{4}{9}\left(c_2 + \frac{1}{2}c_1^2\right)$$

$$a_4 = \frac{1}{3}(c_1c_2 + c_3)$$

$$a_5 = \frac{2}{15}\left[2(c_4 + c_1c_3) + c_2^2\right]$$
$$a_6 = \frac{2}{9}\left[(c_5 + c_1c_4) + c_2c_3\right]$$

and applying triangle inequalities and the inequalities (2.1) and (2.2), we have the desired bounds.

Next, from (3.3) to (3.5), we have

$$a_3a_5 - a_4^2 = \frac{1}{135}\left(16c_2c_4 + 8c_1^3c_3 + 8c_1^2c_4 - 11c_1^2c_2^2 - 14c_1c_2c_3 + 8c_2^3 - 15c_3^2\right)$$

Rearranging, we have

$$a_3a_5 - a_4^2 = \frac{1}{135}\left(8c_2c_4 + 8c_2^3 + 8c_2c_4 - 8c_1c_2c_3 - 6c_3^2 - 6c_1c_2c_3 \right.$$
$$\left. -9c_3^2 + 9c_1^3c_3 + 8c_1^2c_4 - 8c_1^2c_2^2 - c_1^3c_3 - c_1^2c_2^2 - 2c_1^2c_2^2\right).$$

Applying triangle inequality, we have

$$|a_3a_5 - a_4^2| \le \frac{1}{135}\left(8|c_2||c_4 + c_2^2| + 8|c_2||c_4 - c_1c_3| + 6|c_3||c_3 - c_1c_2| \right.$$
$$\left. +9|c_3||c_3 - c_1^3| + 8|c_1|^2|c_4 - c_2^2| + |c_1|^2|c_1c_3 - c_2^2| + 2|c_1|^2|c_2|^2\right).$$

By applying the inequalities (2.2) to (2.4) and (2.8), we have

$$|a_3a_5 - a_4^2| \le \frac{1}{135}(16|c_2| + 15|c_3| + 9|c_1|^2 + 2|c_1|^2|c_2|^2).$$

Substituting for $|c_2|$ and $|c_3|$ using the inequalities (2.5) and (2.6), then rearranging, we have:

$$|a_3a_5 - a_4^2| \le \frac{1}{135}\left[31 - 20|c_1|^2 - 2|c_1|^4 - 15\frac{|c_2|^2}{1+|c_1|}\right].$$

Setting $x = |c_1|$ and $y = |c_2|$ we have:

$$|a_3a_5 - a_4^2| \le \frac{1}{135}h(x,y)$$

where

$$h(x,y) = 31 - 20x^2 - 2x^4 - 15\frac{y^2}{1+x}.$$

Noting $y = |c_2| \ge 0$, we have

$$h(x,y) \le h(x,0) = 31 - 20x^2 - 2x^4.$$

Similarly, since $x = |c_1| \ge 0$ it follows that the maximum of $h(x,0)$ is at $x = 0$ and its maximum value is 31. Hence we have $h(x,y) \le h(x,0) \le h(0,0) = 31$ as required.

Thus, we have

$$|a_3a_5 - a_4^2| \le \frac{31}{135}.$$

Next, we have from (3.2) to (3.5)

$$a_3a_4 - a_2a_5 = \frac{1}{135}\left[20c_2c_3 + 10c_1^3c_2 + 8c_1c_2^2 - 14c_1^2c_3 + 24c_1c_4\right].$$

Rearranging, we have

$$a_3a_4 - a_2a_5 = \frac{1}{135}\left[18c_2c_3 + 18c_1^3c_2 + 2c_2c_3 + 2c_1c_2^2 + 6c_1c_2^2 - 6c_1^3c_2 \right.$$
$$\left. -2c_1^3c_2 - 2c_1^2c_3 - 12c_1^2c_3 + 12c_1c_4 + 12c_1c_4\right]$$

and by triangle inequality, we have

$$|a_3a_4 - a_2a_5| \leq \frac{1}{135}\left[18|c_2||c_3 + c_1^3| + 2|c_2||c_3 + c_1c_2| + 6|c_1||c_2||c_2 - c_1^2| \right.$$
$$\left. +2|c_1|^2|c_1c_2 + c_3| + 12|c_1||c_4 - c_1c_3| + 12|c_1||c_4|\right].$$

Now applying the inequalities (2.1) to (2.4) and (2.8) we have

$$|a_3a_4 - a_2a_5| \leq \frac{52}{135}.$$

Then next, we have from (3.2) to (3.4)

$$a_2a_4 - a_3^2 = \frac{2}{81}c_1^2c_2 - \frac{4}{81}c_1^4 + \frac{2}{9}c_1c_3 - \frac{16}{81}c_2^2$$

which gives

$$a_2a_4 - a_3^2 = \frac{2}{81}c_1^2c_2 - \frac{2}{81}c_1^4 - \frac{2}{81}c_1^4 + \frac{2}{81}c_1c_3 + \frac{16}{81}c_1c_3 - \frac{16}{81}c_2^2.$$

which gives

$$|a_2a_4 - a_3^2| \leq \frac{2}{81}|c_1^2||c_2 - c_1^2| + \frac{2}{81}|c_1||c_3 - c_1^3| + \frac{16}{81}|c_1c_3 - c_2^2|.$$

Now applying inequalities (2.1) to (2.4) and (2.8) we have:

$$|a_2a_4 - a_3^2| \leq \frac{20}{81}.$$

Finally, using the above results, we conclude that $|H_3(2)|$ given by

$$|H_3(2)| \leq |a_4||a_3a_5 - a_4^2| + |a_5||a_3a_4 - a_2a_5| + |a_6||a_2a_4 - a_3^2|$$

satisfies the inequality:

$$|H_3(2)| \leq \frac{6203}{18225} = 0.340356652949246.$$

This concludes the proof. □

Remark 3.1. In the paper [2], the authors presented the following lemma neither with a proof nor a citation referencing a source!

Lemma 3.2. *. If $w(z) = c_1z + c_2z^2 + \cdots$ is a Schwarz function and μ and v are real numbers satisfying*

$$\frac{1}{2} \leq |\mu| \leq 2,\ \frac{4}{27}(|\mu| + 1)^3 - (|\mu| + 1) \leq v \leq 1,$$

then

$$|c_3 + \mu c_1c_2 + vc_1^3| \leq 1.$$

The present authors have tried to search through databases of journals of mathematical analysis for this result without success! We did not stop at that. The first author has also reached out to the authors of [2] via emails also without success. While not outrightly disapproving the assertion of the lemma, it appears to the present authors more of a conjecture than an established result, except for some specific μ and v for which it is true.

In view of this, we have refrained from employing the lemma in this work. We however wish it were true, it would have been of immense benefit in improving the estimates of the bounds on functionals of the types being considered in this study.

In fact, employing this lemma indirectly, they estimated that $|a_2a_4-a_3^2| \leq \frac{2}{9} = \frac{18}{81}$. However, the correct estimate is $|a_2a_4 - a_3^2| \leq \frac{20}{81}$ we have reported above.

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